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New analyticity constraints on the high energy behavior of hadron–hadron cross sections

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Abstract. We here comment on a series of recent papers by Igi and Ishida and Block and Halzen that fit high energy pp and $\bar{p}p$ cross section and ρ-value data, where ρ is the ratio of the real to the imaginary portion of the forward scattering amplitude. These authors used finite-energy sum rules and analyticity consistency conditions, respectively, to constrain the asymptotic behavior of hadron cross sections by anchoring their high energy asymptotic amplitudes – even under crossing – to low energy experimental data. Using analyticity, we here show that i) the two apparently very different approaches are in fact equivalent, ii) that these analyticity constraints can be extended to give new constraints, and iii) that these constraints can be extended to crossing-odd amplitudes. We also apply these extensions to photoproduction. A new interpretation of duality is given.

About 40 years ago, Dolen, Horn and Schmid [1] used analyticity to derive finite-energy sum rules, FESRs, to determine Regge parameters (for what were then high energies) from low energy data. Very recently, Igi and Ishida, again using analyticity, developed FESRs for both pion–proton scattering [2] and for pp and $\bar{p}p$ scattering [3] for rising cross sections at present day energies. They exploited the very precise experimental cross section information, $\sigma_{\rm tot}(pp)$ and $\sigma_{\rm tot}(\bar{p}p)$, available for *low energy* scattering, to constrain the coefficients of a real analytic amplitude fit they made to the even (under crossing) cross section $\sigma_+(\nu)$ at *high energies*. Block and Halzen [4, 5], taking a very different approach, required that both the hh (hadron–hadron) and the $\bar{h}h$ low energy cross sections constrain the high energy fit, using

$$\sigma_{\rm tot}(\nu_0)=\tilde{\sigma}(\nu_0) \qquad \text{and} \qquad \frac{{\rm d}\sigma_{\rm tot}}{{\rm d}\nu}(\nu_0)=\frac{{\rm d}\tilde{\sigma}}{{\rm d}\nu}(\nu_0)\,,$$

where $\sigma_{\rm tot}(\nu_0)$ is the *experimental* hh or $\bar{h}h$ total cross section at laboratory energy ν_0 and $\tilde{\sigma}(\nu_0)$ is the total cross section at ν_0 obtained from the high energy parameterization that was used to fit the high energy hh or $\bar{h}h$ cross section data for hadron–hadron scattering; both even and odd amplitudes (under crossing) were used. In the above, the transition energy ν_0 was chosen to be an energy just above the resonance region, where the cross section energy dependence is smooth and featureless. In particular, they successfully fit γp [4] and separately, π^+p, π^-p and pp, $\bar{p}p$ scattering [5] with a $\ln^2 s$ parameterization. In a separate work [6], they showed that they got identical numerical results using these constraints as they got from using the Igi and Ishida constraint [3], when fitting the same data set of pp and $\bar{p}p$ high energy cross sections. We will show below that the two approaches are equivalent, with both following from analyticity requirements. In deriving their FESR(2) for pp and $\bar{p}p$ scattering [3], Igi and Ishida [3] took a slightly different philosophy from Dolen, Horn and Schmid [1] in that they used terms for the high energy behavior that involved non-Regge amplitudes such as terms in $\ln s$ and $\ln^2 s$, in addition to the Regge poles of [1]. They chose for their crossing-even high energy forward scattering amplitude[1] $\tilde{f}_+(\nu)$ (we have $\tilde{f}_+(-\nu)=\tilde{f}_+(\nu)$)

$$\mathrm{Im}\tilde{f}_+(\nu)=\frac{\nu}{m^2}\left[C_0+C_1\ln\left(\frac{\nu}{m}\right)+C_2\ln^2\left(\frac{\nu}{m}\right)+B_{\mathcal{P}'}\left(\frac{\nu}{m}\right)^{\mu-1}\right], \tag{1}$$

$$\mathrm{Re}\tilde{f}_+(\nu)=\frac{\nu}{m^2}\left[\frac{\pi}{2}C_1+C_2\pi\ln\left(\frac{\nu}{m}\right)-B_{\mathcal{P}'}\cot\left(\frac{\pi\mu}{2}\right)\left(\frac{\nu}{m}\right)^{\mu-1}\right], \tag{2}$$

where m is the proton mass and ν is the laboratory projectile energy, with real dimensionless coefficients $C_0,\ C_1,\ C_2$ and $B_{\mathcal{P}'}$.

[1] We have changed their notation slightly, replacing the amplitude F by f, and the energy N by ν_0. In what follows, m is the proton mass, p is the laboratory momentum and ν is the laboratory energy. We have changed their notation for their dimensionless parameters, letting $c_0\to C_0, c_1\to C_1, c_2\to C_2$ and $\beta_{\mathcal{P}'}\to B_{\mathcal{P}'}$.

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We comment that had they used the factor p/m^2 rather than ν/m^2 in front of the right-hand sides of (1) and (2) as required by analyticity for a real even amplitude, their choice of amplitude would have been an even real analytic function and $f_+(\nu)$ would be zero for $0 \le \nu \le m$, the proton mass, as required for a real analytic forward scattering amplitude [7]. In the high energy limit – in (1) and (2) – they replaced the laboratory momentum $p = \sqrt{\nu^2 - m^2}$ by ν. Using the optical theorem, after letting $p \to \nu$, they obtained the even cross section from (1) as

$$\tilde{\sigma}_+(\nu) = \frac{4\pi}{m^2}\left[C_0 + C_1 \ln(\nu/m) + C_2 \ln^2(\nu/m) + B_{\mathcal{P}'}(\nu/m)^{\mu-1}\right], \quad (3)$$

valid in the high energy region $\nu \gtrsim \nu_0$. They used a Reggion trajectory with $\mu = 0.5$.

Block and Cahn [5] used a similar parameterization to analyze pp and $\bar{p}p$ cross sections and ρ-values. Their even real analytic forward high energy scattering amplitude $\tilde{f}_+(\nu)$ is given by:

$$\mathrm{Im}\tilde{f}_+(\nu) = \frac{p}{4\pi}\left[c_0 + c_1 \ln\left(\frac{\nu}{m}\right) + c_2 \ln^2\left(\frac{\nu}{m}\right) + \beta_{\mathcal{P}'}\left(\frac{\nu}{m}\right)^{\mu-1}\right] \quad \text{for } \nu \ge m\,,$$
$$\mathrm{Im}\tilde{f}_+(\nu) = 0 \qquad \text{for } 0 \le \nu \le m\,, \quad (4)$$
$$\mathrm{Re}\tilde{f}_+(\nu) = \frac{p}{4\pi}\left[\frac{\pi}{2}c_1 + c_2\pi \ln\left(\frac{\nu}{m}\right) - \beta_{\mathcal{P}'}\cot\left(\frac{\pi\mu}{2}\right)\left(\frac{\nu}{m}\right)^{\mu-1}\right]. \quad (5)$$

Using the optical theorem, their even cross section is

$$\tilde{\sigma}_+(\nu) = c_0 + c_1 \ln(\nu/m) + c_2 \ln^2(\nu/m) + \beta_{\mathcal{P}'}(\nu/m)^{\mu-1}\,, \quad (6)$$

where here the coefficients c_0, c_1, c_2 and $\beta_{\mathcal{P}'}$ have dimensions of mb.

We now introduce $f_+(\nu)$, the true even forward scattering amplitude (which of course, we do not know!), valid for all ν, where $f_+(\nu) \equiv [f_{pp}(\nu) + f_{\bar{p}p}(\nu)]/2$, using forward scattering amplitudes for pp and $\bar{p}p$ collisions. Using the optical theorem, the imaginary portion of $f_+(\nu)$ is related to the even total cross section $\sigma_{\mathrm{even}}(\nu)$ by

$$\mathrm{Im}f_+(\nu) = \frac{p}{4\pi}\sigma_{\mathrm{even}}(\nu)\,, \qquad \text{for } \nu \ge m\,. \quad (7)$$

Next, define the odd amplitude $\nu\hat{f}_+(\nu)$ as the difference

$$\nu\hat{f}_+(\nu) \equiv \nu\left[f_+(\nu) - \tilde{f}_+(\nu)\right], \quad (8)$$

which satisfies the unsubtracted odd amplitude dispersion relation

$$\mathrm{Re}\nu\hat{f}_+(\nu) = \frac{2\nu}{\pi}\int_0^\infty \frac{\mathrm{Im}\nu'\hat{f}_+(\nu')}{\nu'^2 - \nu^2}\,\mathrm{d}\nu'\,. \quad (9)$$

Since for large ν, the odd amplitude $\nu\hat{f}_+(\nu) \sim \nu^\alpha$ ($\alpha < 0$) by design, it also satisfies the super-convergence relation

$$\int_0^\infty \mathrm{Im}\,\nu\hat{f}_+(\nu)\,\mathrm{d}\nu = 0\,. \quad (10)$$

In [1], the FESRs are given by

$$\int_0^{\nu_0} \nu^n\,\mathrm{Im}\hat{f}\,\mathrm{d}\nu = \sum \frac{{\nu_0}^{\alpha+n+1}}{\alpha+n+1}, \qquad n = 0, 1, \ldots, \infty\,, \quad (11)$$

where $\hat{f}(\nu)$ is crossing-even for odd integer n and crossing-odd for even integer n. In analogy to the $n = 1$ FESR of [1], which requires the odd amplitude $\nu\hat{f}(\nu)$, Igi and Ishida inserted the super-convergent amplitude of (8) into the super-convergent dispersion relation of (10), obtaining

$$\int_0^\infty \nu\,\mathrm{Im}\left[f_+(\nu) - \tilde{f}_+(\nu)\right]\,\mathrm{d}\nu\,. \quad (12)$$

We note that the odd difference amplitude $\nu\mathrm{Im}\hat{f}_+(\nu)$ satisfies (10), a super-convergent dispersion relation, even if neither $\nu\,\mathrm{Im}f_+(\nu)$ nor $\nu\,\mathrm{Im}\tilde{f}_+(\nu)$ satisfies it. Since the integrand of (12), $\nu\,\mathrm{Im}\left[f_+(\nu) - \tilde{f}_+(\nu)\right]$, is super-convergent, we can truncate the upper limit of the integration at the finite energy ν_0, an energy high enough for resonance behavior to vanish and where the difference between the two amplitudes – the true amplitude $f_+(\nu)$ minus $\tilde{f}_+(\nu)$, the amplitude which parameterizes the high energy behavior – becomes negligible, so that the integrand can be neglected for energies greater than ν_0. Thus, after some rearrangement, we get the even finite-energy sum rule (FESR)

$$\int_0^{\nu_0} \nu\mathrm{Im}f_+(\nu)\,\mathrm{d}\nu = \int_0^{\nu_0} \nu\mathrm{Im}\tilde{f}_+(\nu)\,\mathrm{d}\nu\,. \quad (13)$$

Next, the left-hand integral of (13) is broken up into two parts, an integral from 0 to m (the 'unphysical' region) and the integral from m to ν_0, the physical region. We use the optical theorem to evaluate the left-hand integrand for $\nu \ge m$. After noting that the imaginary portion of $\tilde{f}_+(\nu) = 0$ for $0 \le \nu \le m$, we again use the optical theorem to evaluate the right-hand integrand, finally obtaining the finite-energy sum rule FESR(2) of Igi and Ishida, in the form:

$$\int_0^m \nu\,\mathrm{Im}f_+(\nu)\,\mathrm{d}\nu + \frac{1}{4\pi}\int_m^{\nu_0} \nu p\,\sigma_{\mathrm{even}}(\nu)\,\mathrm{d}\nu = \frac{1}{4\pi}\int_m^{\nu_0} \nu p\,\tilde{\sigma}_+(\nu)\,\mathrm{d}\nu\,. \quad (14)$$

We now enlarge on the consequences of (14). We note that if (14) is valid at the upper limit ν_0, it certainly is also valid at $\nu_0 + \Delta\nu_0$, where $\Delta\nu_0$ is very small compared to ν_0, i.e., $0 \le \Delta\nu_0 \ll \nu_0$. Evaluating (27) at the energy $\nu_0 + \Delta\nu_0$ and then subtracting (14) evaluated at ν_0, we find

$$\frac{1}{4\pi}\int_{\nu_0}^{\nu_0+\Delta\nu_0} \nu p\,\sigma_{\mathrm{even}}(\nu)\,\mathrm{d}\nu = \frac{1}{4\pi}\int_{\nu_0}^{\nu_0+\Delta\nu_0} \nu p\tilde{\sigma}_+(\nu)\,\mathrm{d}\nu\,. \quad (15)$$

Clearly, in the limit of $\Delta\nu_0 \to 0$, (15) goes into

$$\sigma_{\rm even}(\nu_0) = \tilde{\sigma}_+(\nu_0)\,. \qquad (16)$$

Obviously, (16) also implies that

$$\sigma_{\rm even}(\nu) = \tilde{\sigma}_+(\nu) \qquad \text{for all}\, \nu \geq \nu_0\,, \qquad (17)$$

but is most useful in practice when ν_0 is as low as possible. The utility of (17) becomes evident when we recognize that the left-hand side of it can be evaluated using the very accurate low energy *experimental* crossing-even total cross section data, whereas the right-hand side can use the phenomenologist's parameterization of the *high energy* cross section. For example, we could use the cross section parameterization of (6) on the right-hand side of (17) and write the constraint

$$[\sigma_{pp}(\nu) + \sigma_{\bar{p}p}(\nu)]\,/2 = c_0 + c_1 \ln(\nu/m) + c_2 \ln^2(\nu/m) + \beta_{\mathcal{P}'}(\nu/m)^{\mu-1}\,, \qquad (18)$$

where σ_{pp} and $\sigma_{\bar{p}p}(\nu)$ are the experimental pp and $\bar{p}p$ cross sections at the laboratory energy ν. Equation (16) (or (17)) is our first important extension, giving us an analyticity constraint, a consistency condition that the even high energy (asymptotic) amplitude must satisfy.

Reiterating, (17) is a consistency condition imposed by analyticity that states that we must fix the even high energy cross section evaluated at energy $\nu \geq \nu_0$ (using the asymptotic even amplitude) to the low energy experimental even cross section at the *same* energy ν, where ν_0 is an energy just above the resonances. Clearly, (16) also implies that all derivatives of the total cross sections match, as well as the cross sections themselves, i.e.,

$$\frac{\mathrm{d}^n\sigma_{\rm even}}{\mathrm{d}\nu^n}(\nu) = \frac{\mathrm{d}^n\tilde{\sigma}_+}{\mathrm{d}\nu^n}(\nu), \quad n = 0, 1, 2, \ldots \qquad \nu \geq \nu_0\,, \qquad (19)$$

giving new even amplitude analyticity constraints. Of course, the evaluation of (19) for $n = 0$ and $n = 1$ is effectively the same as evaluating (19) for $n = 0$ at two nearby values, ν_0 and $\nu_1 > \nu_0$. It is up to the phenomenologist to decide which experimental set of quantities it is easier to evaluate.

We emphasize that these consistency constraints are the consequences of imposing analyticity, implying several important conditions.

1. The new constraints that are derived here tie together both the even hh and $\bar{h}h$ experimental cross sections and their derivatives to the even high energy approximation that is used to fit data at energies well above the resonance region. Analyticity then requires that there should be a *good* fit to the high energy data *after* using these constraints, i.e., the χ^2 per degree of freedom of the constrained fit should be ~ 1, if the high energy asymptotic amplitude is a good approximation to the high energy data. This is our consistency condition demanded by analyticity. If, on the other hand, the high energy asymptotic amplitude would have given a somewhat poorer fit to the data when *not* using the new constraints, the effect is tremendously magnified by utilizing these new constraints, yielding a very large χ^2 per degree of freedom. As an example, both Block and Halzen [4] and Igi and Ishida [2, 3] conclusively rule out a $\ln s$ fit to both $\pi^{\pm}p$ and pp and $\bar{p}p$ cross sections and ρ-values because it has a huge χ^2 per degree of freedom.
2. Consistency with analyticity requires that the results be valid for all $\nu \geq \nu_0$, so that the constraint doesn't depend on the particular choice of ν.
3. No evaluation of the non-physical integral $\int_0^m \nu\, \mathrm{Im} f_+(\nu)\, \mathrm{d}\nu$ used in (14) is needed for our new constraints. Thus, the exact value of non-physical integrals, even if very large, does not affect our new constraints.
4. As stated before, ν_0 is an energy slightly above the resonance region where the energy behavior of the cross section is smooth and featureless. Duality previously has been used to state that the average value of the energy moments of the imaginary portion of the true amplitude over the energy interval 0 to ν_0 are the same as the average value of the energy moments of the high energy approximation amplitude over the same interval [1], which is illustrated in Fig. 1.
Here, we present a new interpretation of duality – we have demonstrated that analyticity requires that the even cross sections and their derivatives deduced from the even dual high energy amplitude at energy ν_0 are the same as those cross section and their derivatives found from low energy experimental cross section data at ν_0, under the caveat that the dual amplitude gives a good representation of the high energy data. Later, we will demonstrate that this is also true for odd amplitudes, so that our new duality interpretation is true for hh and $\bar{h}h$ cross sections as well.

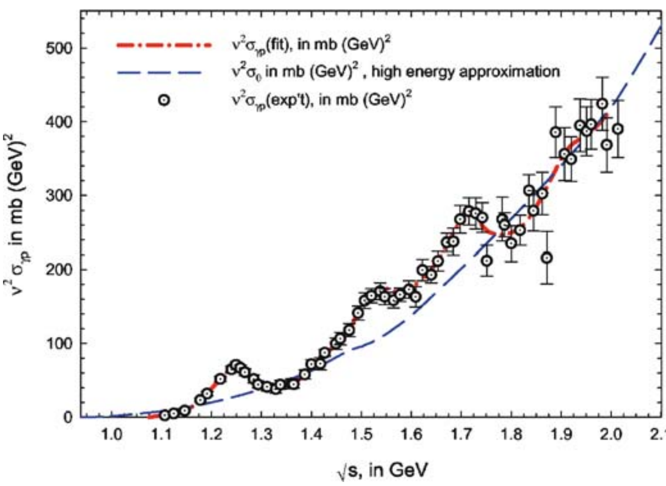

Fig. 1. The integrands of the FESR2 rule. The open circles are $\nu^2 \times \sigma_{\gamma p}$(exp't), the dash-dotted curve is $\nu^2 \times \sigma_{\gamma p}$(fit), and the *dashed curve* is $\nu^2 \times \sigma_0$, all in mb (GeV)^2, versus $\sqrt{s}$, in GeV. $\sigma_0 = c_0 + c_1 \ln(\nu/m) + \ln^2(\nu/m) + \beta_{\mathcal{P}'}(\nu/m)^{-.5}$ is the theoretical high-energy fit of Block and Halzen [4], $\sigma_{\gamma p}$(fit) is the resonance cross section fit of Damashek and Gilman [9], and $\sigma_{\gamma p}$(exp't) are the experimental data in the resonance region. The transition energy was $\sqrt{s_0} = 2.01$ GeV

Having restricted ourselves so far to even amplitudes, let us now consider odd amplitudes, defining $f_-(\nu)$ as the true odd forward scattering amplitude, valid for all ν (again, which we do not know!). In terms of the forward scattering amplitudes for pp and $\bar{p}p$ collisions, we define $f_-(\nu) \equiv [f_{pp}(\nu) - f_{\bar{p}p}(\nu)]/2$. Using the optical theorem, the imaginary portion of the odd amplitude is related to the physical odd (under crossing) total cross section $\sigma_{\rm odd}$ by

$$\mathrm{Im} f_-(\nu) = \frac{p}{4\pi}\sigma_{\rm odd}(\nu), \qquad \text{for } \nu \geq m\,. \tag{20}$$

We now define a new super-convergent odd amplitude $\hat{f}_-(\nu)$ as

$$\hat{f}_-(\nu) \equiv f_-(\nu) - \tilde{f}_-(\nu)\,, \tag{21}$$

where $\tilde{f}_-(\nu)$ is our high energy parameterization amplitude, related to the odd (under crossing) high energy cross section $\sigma_-(\nu)$ by

$$\begin{aligned} \mathrm{Im}\tilde{f}_-(\nu) &= \frac{p}{4\pi}\sigma_-(\nu), \quad \text{for } \nu \geq m\,,\\ \mathrm{Im}\tilde{f}_-(\nu) &= 0\,, \qquad\qquad \text{for } 0 \leq \nu \leq m\,. \end{aligned} \tag{22}$$

The super-convergent amplitude of (21) satisfies the unsubtracted odd amplitude dispersion relation

$$\hat{f}_-(\nu) = \frac{2\nu}{\pi}\int_0^\infty \frac{\mathrm{Im}\,\hat{f}_-(\nu')}{\nu'^2 - \nu^2}\,\mathrm{d}\nu'\,, \tag{23}$$

and, as before, it also satisfies the super-convergent dispersion relation

$$\int_0^\infty \mathrm{Im}\,\hat{f}_-(\nu)\,\mathrm{d}\nu = 0\,. \tag{24}$$

Again, we can truncate the integral at ν_0, so that

$$\int_0^{\nu_0} \mathrm{Im}\,\hat{f}_-(\nu)\,\mathrm{d}\nu = 0\,, \tag{25}$$

or

$$\int_0^{\nu_0} \mathrm{Im}\,f_-(\nu)\,\mathrm{d}\nu = \int_0^{\nu_0} \mathrm{Im}\,\tilde{f}_-(\nu)\,\mathrm{d}\nu\,. \tag{26}$$

After applying the optical theorem, using (20) on the left-hand side and (22) on the right-hand side of (26), we write our new $n=0$ odd finite-energy sum rule, called FESR(odd), as

$$\begin{aligned} &\int_0^m \mathrm{Im} f_-(\nu)\,\mathrm{d}\nu + \frac{1}{4\pi}\int_m^{\nu_0} p\sigma_{\rm odd}(\nu)\,\mathrm{d}\nu \\ &= \frac{1}{4\pi}\int_m^{\nu_0} p\tilde{\sigma}_-\,\mathrm{d}\nu\,. \end{aligned} \tag{27}$$

Following the same line as before, it is straightforward to show for odd amplitudes that FESR(odd) implies that

$$\frac{\mathrm{d}^n\sigma_{\rm odd}}{\mathrm{d}\nu^n}(\nu) = \frac{\mathrm{d}^n\tilde{\sigma}_-}{\mathrm{d}\nu^n}(\nu), \quad n = 0,1,2,\dots\,, \qquad \nu \geq \nu_0\,, \tag{28}$$

where $\tilde{\sigma}_-(\nu)$ is the odd (under crossing) high energy cross section approximation and $\sigma_{\rm odd}(\nu)$ is the experimental odd cross section.

Thus, we have now derived new analyticity constraints for *both* even and odd cross sections, allowing us to constrain both hh and $\bar{h}h$ scattering. All of the conditions 1–4, enumerated earlier for even amplitudes, are now valid for odd amplitudes, and hence, for both hh and $\bar{h}h$ scattering.

Block and Halzen [5] expanded upon these ideas, using linear combinations of cross sections and derivatives to anchor *both* even and odd cross sections. A total of 4 constraints, 2 even and 2 odd constraints, were used by them in their successful $\ln^2 s$ fit to pp and $\bar{p}p$ cross sections and ρ-values, where they first did a local fit to pp and $\bar{p}p$ cross sections and their slopes in the neighborhood of $\nu_0 = 7.59$ GeV (corresponding to $\sqrt{s_0} = 4$ GeV), to determine the experimental cross sections and their first derivatives at which they anchored their fit. The data they used in the high energy fit were pp and $\bar{p}p$ cross sections and ρ-values with energies $\sqrt{s} \geq 6$ GeV. Introducing the even cross section $\sigma_0(\nu)$, they parameterized the high energy cross sections and ρ-values [5] as

$$\sigma_0(\nu) = c_0 + c_1\ln\left(\frac{\nu}{m}\right) + c_2\ln^2\left(\frac{\nu}{m}\right) + \beta_{\mathcal{P}'}\left(\frac{\nu}{m}\right)^{\mu-1}\,, \tag{29}$$

$$\sigma^\pm(\nu) = \sigma_0\left(\frac{\nu}{m}\right) \pm \delta\left(\frac{\nu}{m}\right)^{\alpha-1}\,, \tag{30}$$

$$\begin{aligned} \rho^\pm(\nu) = \frac{1}{\sigma^\pm}\Bigg\{&\frac{\pi}{2}c_1 + c_2\pi\ln\left(\frac{\nu}{m}\right) \\ &- \beta_{\mathcal{P}'}\cot\left(\frac{\pi\mu}{2}\right)\left(\frac{\nu}{m}\right)^{\mu-1} + \frac{4\pi}{\nu}f_+(0) \\ &\pm \delta\tan\left(\frac{\pi\alpha}{2}\right)\left(\frac{\nu}{m}\right)^{\alpha-1}\Bigg\}\,. \end{aligned} \tag{31}$$

We note that the even coefficients c_0, c_1, c_2 and $\beta_{\mathcal{P}'}$ are the same as those used in (18). The real constant $f_+(0)$ is the subtraction constant [7][2] required at $\nu = 0$ for a singly-subtracted dispersion relation. They also used $\mu = 0.5$. The odd cross section in (30) is given by

$$\delta\left(\frac{\nu}{m}\right)^{\alpha-1}\,, \tag{32}$$

described by two parameters, the coefficient δ and the Regge power α. We define

$$\Delta\sigma(\nu_0) \equiv \frac{\sigma^+(\nu_0/m) - \sigma^-(\nu_0/m)}{2} = \delta(\nu_0/m)^{\alpha-1}\,, \tag{33}$$

$$\begin{aligned} \Delta m(\nu_0) &\equiv \frac{1}{2}\left(\frac{\mathrm{d}\sigma^+}{\mathrm{d}(\nu/m)} - \frac{\mathrm{d}\sigma^-}{\mathrm{d}(\nu/m)}\right)_{\nu=\nu_0} \\ &= \delta\left\{(\alpha-1)(\nu_0/m)^{\alpha-2}\right\}\,, \end{aligned} \tag{34}$$

in terms of the odd experimental values at ν_0. Since now δ and α are completely fixed by the experimental quantities $\Delta\sigma(\nu_0)$ and $\Delta m(\nu_0)$, these two analytic constraints

[2] For the reaction $\gamma + p \rightarrow \gamma + p$, it is fixed as the Thompson scattering limit $f_+(0) = -\alpha m = -3.03\,\mu$b GeV [9].

severely restrict the phenomenologist using this particular choice of amplitude. If (32) is not a particularly good representation of the high energy data, the χ^2 from the fit will be very poor. On the other hand, if the χ^2 is very good – as found by Block and Halzen [5] – it provides great confidence in the choice of (32) as the imaginary portion of the asymptotic odd amplitude.

Finally, to get a physical picture of what the new analyticity constraints look like compared to FESR(2), we apply (29), the even cross section, to spin-averaged γp scattering. For the γp system, the left-hand integral of (14), involving experimental cross sections $\sigma_{\gamma p}(\nu)$ in the resonance regions ($\nu \le \nu_0$), is now $\int_0^{\nu_0} \nu^2 \sigma_{\gamma p}(\nu)\,\mathrm{d}\nu$. As a function of the center-of-mass energy $\sqrt{s}$, Fig. 1 shows 3 separate plots: the experimental resonance cross section data multiplied by ν^2 as the open circles; the $\sigma_{\gamma p}(\mathrm{fit})$ – a fit to the resonance region made by Damashek and Gilman [9] – after multiplication by ν^2, as the dashed-dot curve; and, finally, the cross section $\sigma_0 \times \nu^2$, where σ_0 is the even high energy cross section parameterization from Block and Halzen [4], as the dashed curve. Block and Halzen used a transition energy $\nu_0 = 1.68\,\mathrm{GeV}$ ($\sqrt{s}_0 = 2.01\,\mathrm{GeV}$) in their fit, requiring that their fit match the experimental cross section and first derivative at ν_0. Note that the resonance data oscillate about the smooth high energy fit, with the oscillations gradually damping out so that experimental data approach the high energy fit as we near the transition energy of $\sqrt{s}_0 = 2.01\,\mathrm{GeV}$.

In conclusion, we now have new analyticity constraints for both even and odd amplitudes – additional tools for the phenomenologist to use in fitting hadron–hadron total cross sections and ρ-values. In practice, these new analyticity constraints are much simpler to use than FESRs. The fits are anchored by the experimental cross section data near the transition energy ν_0, so neither complicated numerical integrations nor evaluation of unphysical regions are required. These consistency constraints are due to the application of analyticity to finite-energy integrals – the analog of analyticity giving rise to traditional dispersion relations when it is applied to integrals with infinite upper limits – giving us a new interpretation of the duality principle.

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